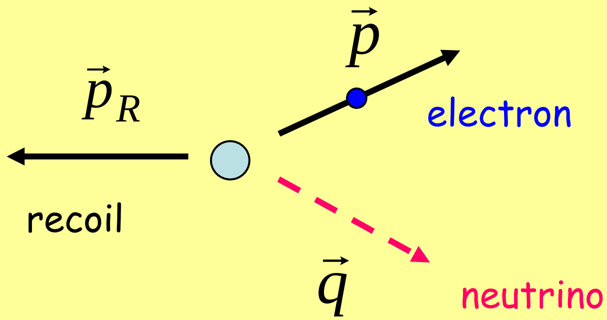
Recall from last class: we calculated the partial decay rate for a final state e- with momentum p using Fermi's Golden Rule:



$$\boxed{\lambda_{if} = \frac{2\pi}{\hbar} \left| M_{if} \right|^2 \rho_f}$$

(transitions / sec)

$$\Rightarrow \quad \lambda_{if} = G^2 \frac{2\pi}{\hbar c} \left| M_{nuclear} \right|^2 \frac{(4\pi)^2}{h^6} p^2 q^2 \, dp$$

where G is a generic "weak coupling constant" for the process.

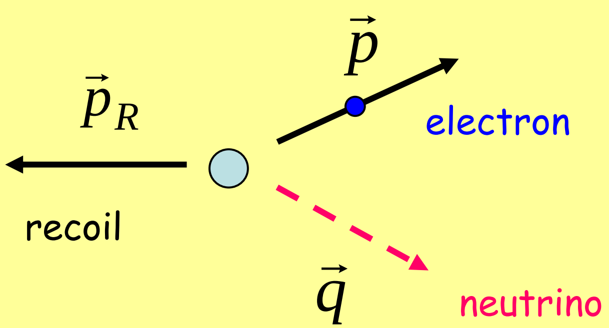
- If the electron and neutrino couple to $S = 0$, then it is called a Fermi decay and $G = G_V$.
- If they couple to $S = 1$, it is a Gamow-Teller decay and $G = G_A$.
- For the neutron decay, both $S = 0$ and $S = 1$ configurations are possible, and $S = 1$ is 3 times more likely, so in that case $G^2 = G_V^2 + 3\, G_A^2$.

Electron momentum spectrum:

$$N(p)\,dp \;=\; N_o\,\lambda_{if} \;=\; (const.)\times p^2 q^2\,dp$$

approx: $K_R = 0$

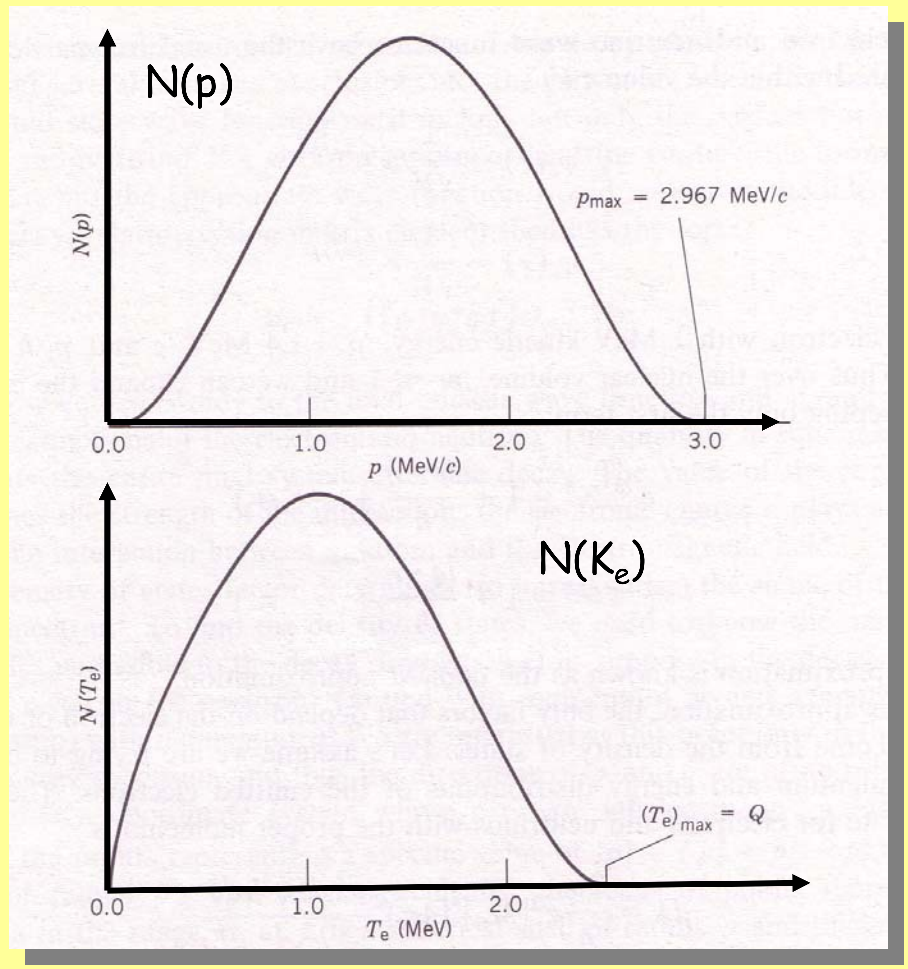
$$\Rightarrow \quad N(p) \;=\; (const.)\times p^2 q^2 \;=\; (const)\;\times p^2\,\overbrace{(Q-K_e)}^{}{}^2$$



Predicted spectral shapes, Krane, figure 9.2:

(plotted for Q = 2.5 MeV, not the neutron!)

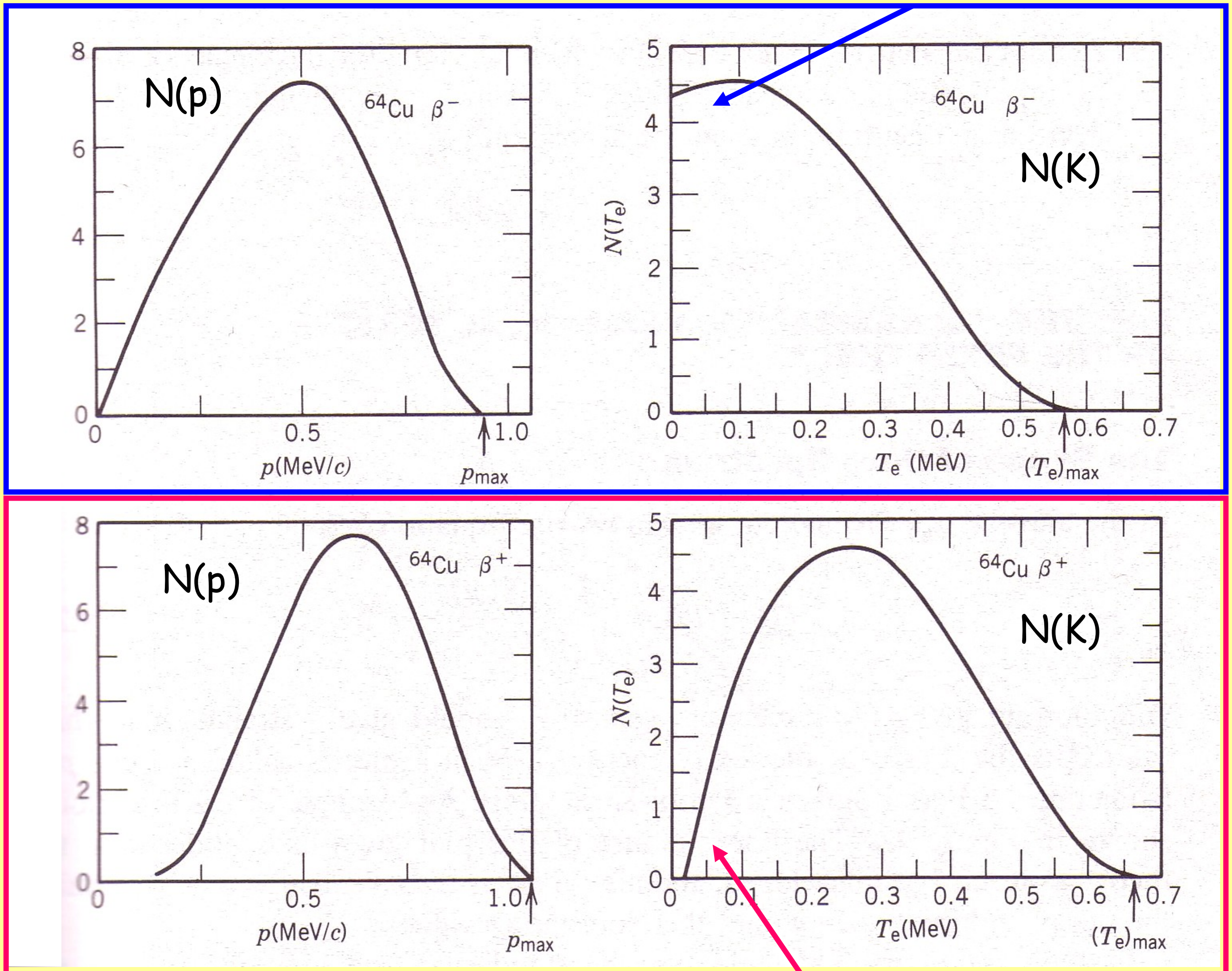
(Note: max. K(e⁻) = Q)
(Krane uses symbol T for K)



Comparison to Reality: (Krane, fig. 9.3: e+ and e- decays of ^{64}Cu)

Too many low energy e^-

???



Coulomb effects ...

Too few low energy e^+

Discrepancy: neglect of Coulomb effects in the final state.

Key point: Coulomb distortions of the energy spectra occur AFTER the electron/positron are emitted in the weak decay process.

Modified density of electron/positron states:

$$dn_e = \left(4\pi\ p^2 dp \frac{V}{h^3} \right) \boxed{F(Z', p)}$$

"**Fermi function**", depends on the charge Z′ of the "daughter nucleus" (final state) and the electron/positron momentum

original result

Approximate correction factor for β± decay:

$$F^{\pm}(Z', p) \cong \frac{x}{1 - e^{-x}}\ , \quad x = \mp \frac{2\pi\,\alpha\, Z' \sqrt{m^2 + p^2}}{p}\ , \quad \alpha = \frac{e^2}{4\pi\varepsilon_o \hbar c} = \frac{1}{137}$$

Modified electron/positron spectrum prediction:

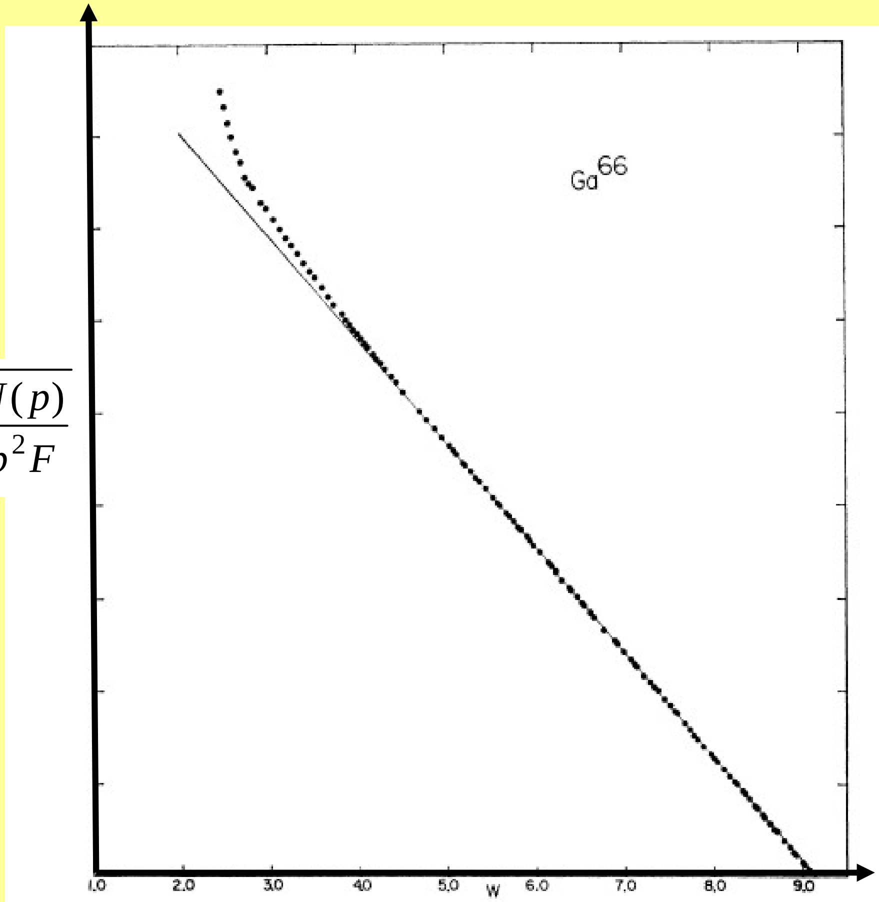
$$N(p) = C\ p^2\ (Q - K_e)^2\ \ F^{\pm}(Z', p)\ , \quad C = \frac{G^2}{2\pi^3 \hbar^7 c^3}\ \left| M_{nucl} \right|^2$$

Fermi-Kurie Plot

Idea: for "allowed decays", corresponding to our approximation: $e^{i\vec{p}_R \bullet \vec{r}/\hbar} = 1$

inside the nucleus, the electron energy spectrum can be "linearized" if one accounts for the Coulomb distortion via the Fermi function F(Z',p):

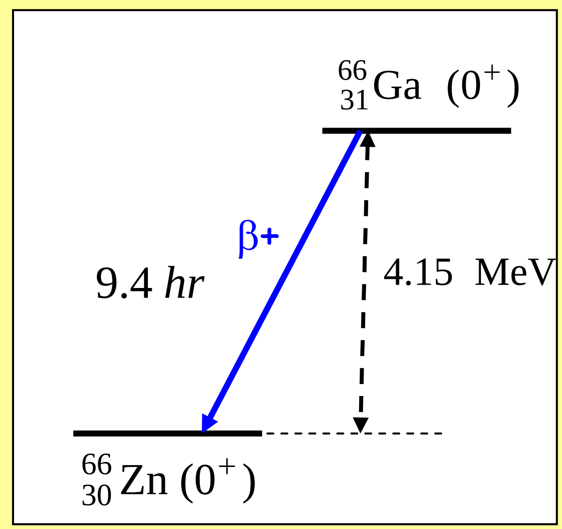
$$\sqrt{\frac{N(p)}{p^2 F}}$$



$$\frac{K_e + m_e}{m_e}$$

$$\sqrt{\frac{N(p)}{p^2 F(Z',p)}} \sim Q - K_e$$

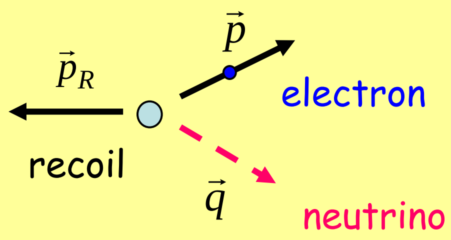
linear function, endpoint Q



$^{66}Ga \rightarrow {}^{66}Zn$ decay, Phys. Rev. ***129****, 1782*

Neutrino Mass effect: $^{3}\mathrm{H} \rightarrow {}^{3}\mathrm{He} + e^{-} + \bar{\nu}_{e} \quad (Q = 18.6 \ \ \mathrm{keV})$

Idea: shape of the electron energy spectrum near the endpoint (Q) is sensitive to the mass of the electron antineutrino:



recall: $Q \equiv K_R + K_e + K_\nu$

When $K_e \cong Q$, $K_R \cong K_\nu \rightarrow 0$. if $m_\nu \neq 0$, then in this limit, mass effects are most pronounced.

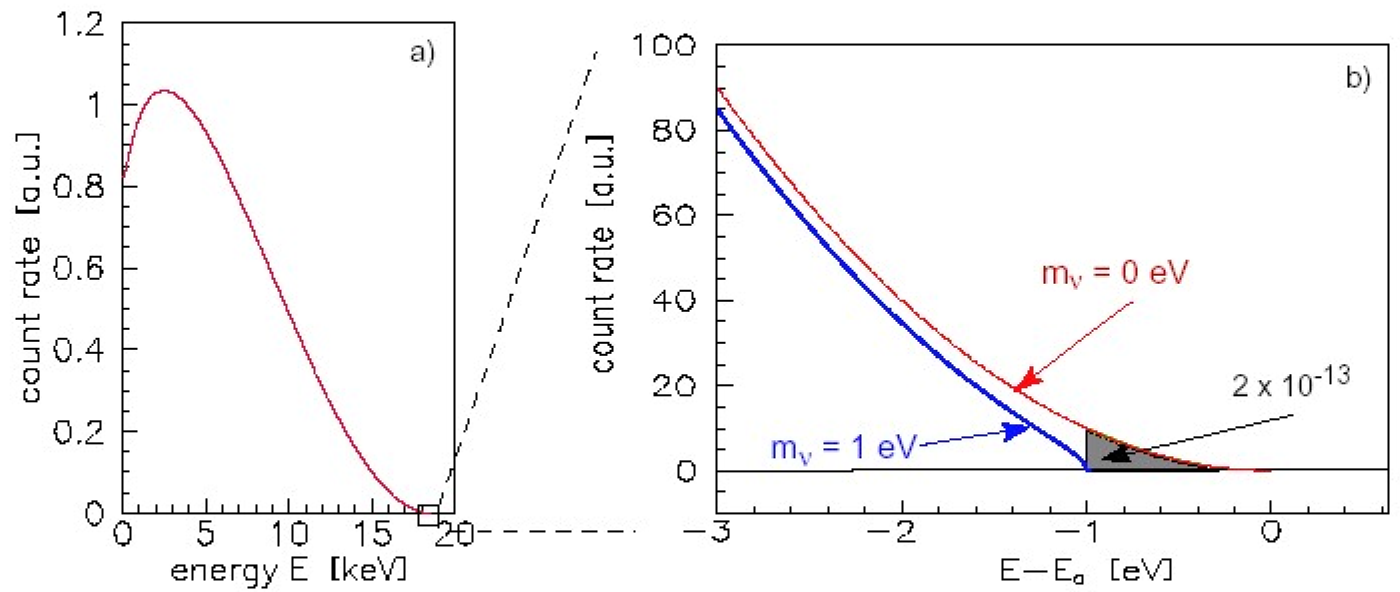
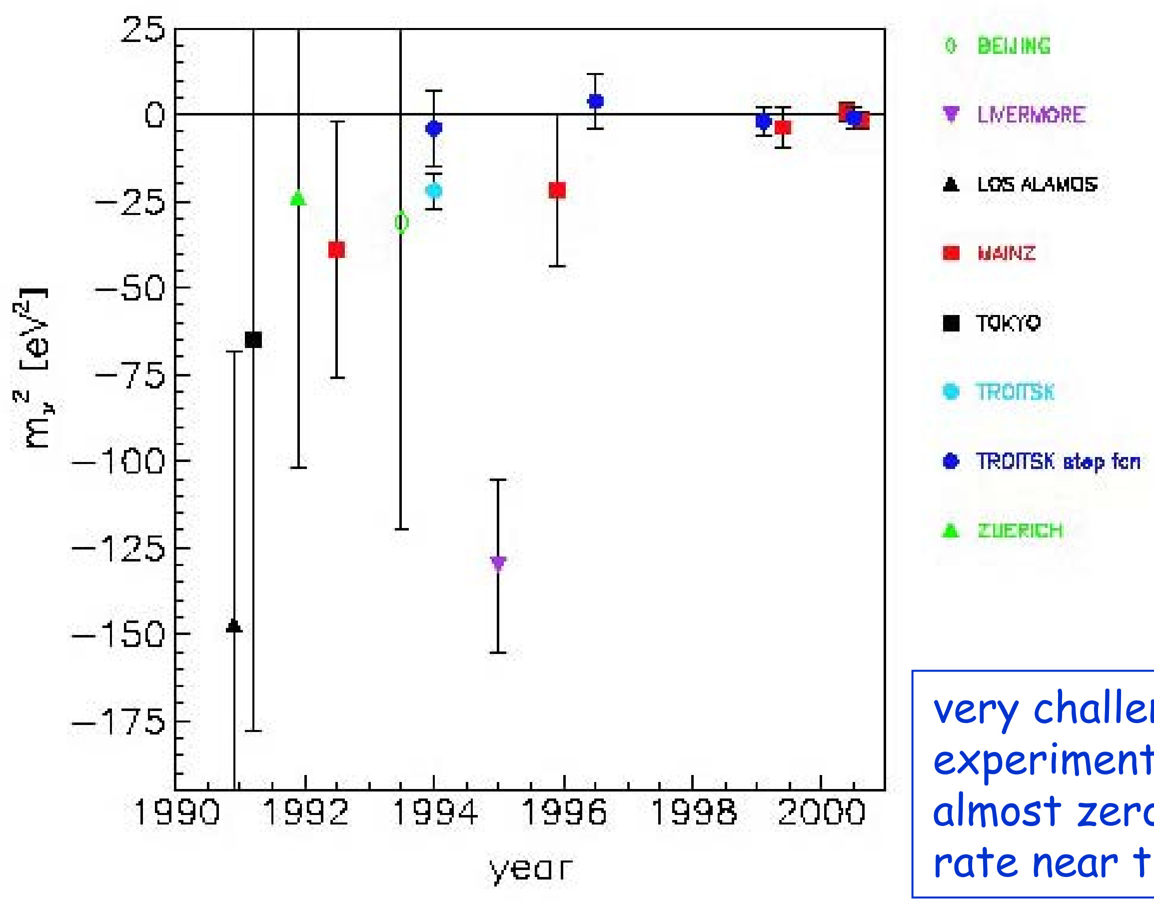


Figure 2: The electron energy spectrum of tritium β decay: (a) complete and (b) narrow region around endpoint E_0. The β spectrum is shown for neutrino masses of 0 and 1 eV.

figure from "KATRIN" proposal, 2001: (Karlsruhe Tritium Neutrino expt.)



very challenging experiments – almost zero count rate near the end point!

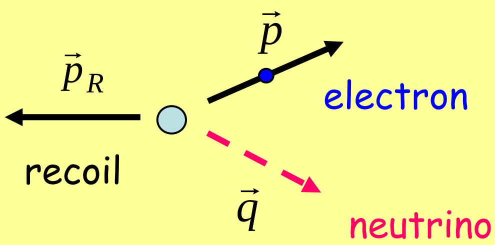
Figure 4: Results of tritium β decay experiments on the observable m_ν^2 over the last decade.

- Best direct upper limit: $m_\nu <$ 2.2 eV
- from Sudbury neutrino observatory and other experiments, we have convincing indirect evidence of nonzero neutrino mass that is much smaller than this

Web sites: http://cupp.oulu.fi/neutrino/nd-mass.html, http://www-ik.fzk.de/~katrin/

Systematic test: total rate for beta decay

Our formalism determines λ_{if}, which is the rate (s^{-1}) to a particular final state electron (or positron) momentum p:



$$\lambda_{if}^{\pm}(p) = \frac{G^2}{2\pi^3\hbar^7 c^3} \left|M_{nucl}\right|^2 p^2 \left(Q - K_e\right)^2 F^{\pm}(Z', p)$$

($\pm$ refers to $\beta^{\pm}$ decay modes)

The total decay rate is obtained by integrating λ_{if} over all allowed $e^{\pm}$ momenta p:

$$\lambda\,(s^{-1}) = \int_0^{p_{\max}} \lambda_{if}(p)\,dp = \frac{G^2}{2\pi^3\hbar^7 c^3} \times \left|M_{nucl}\right|^2 \boxed{\int_0^{p_{\max}} p^2 (Q-K_e)^2 F(Z',p)\,dp}$$

$$= (const.) \times \left|M_{nucl}\right|^2 \boxed{f(Z',Q)}$$

"Fermi integral", f (Z',Q)

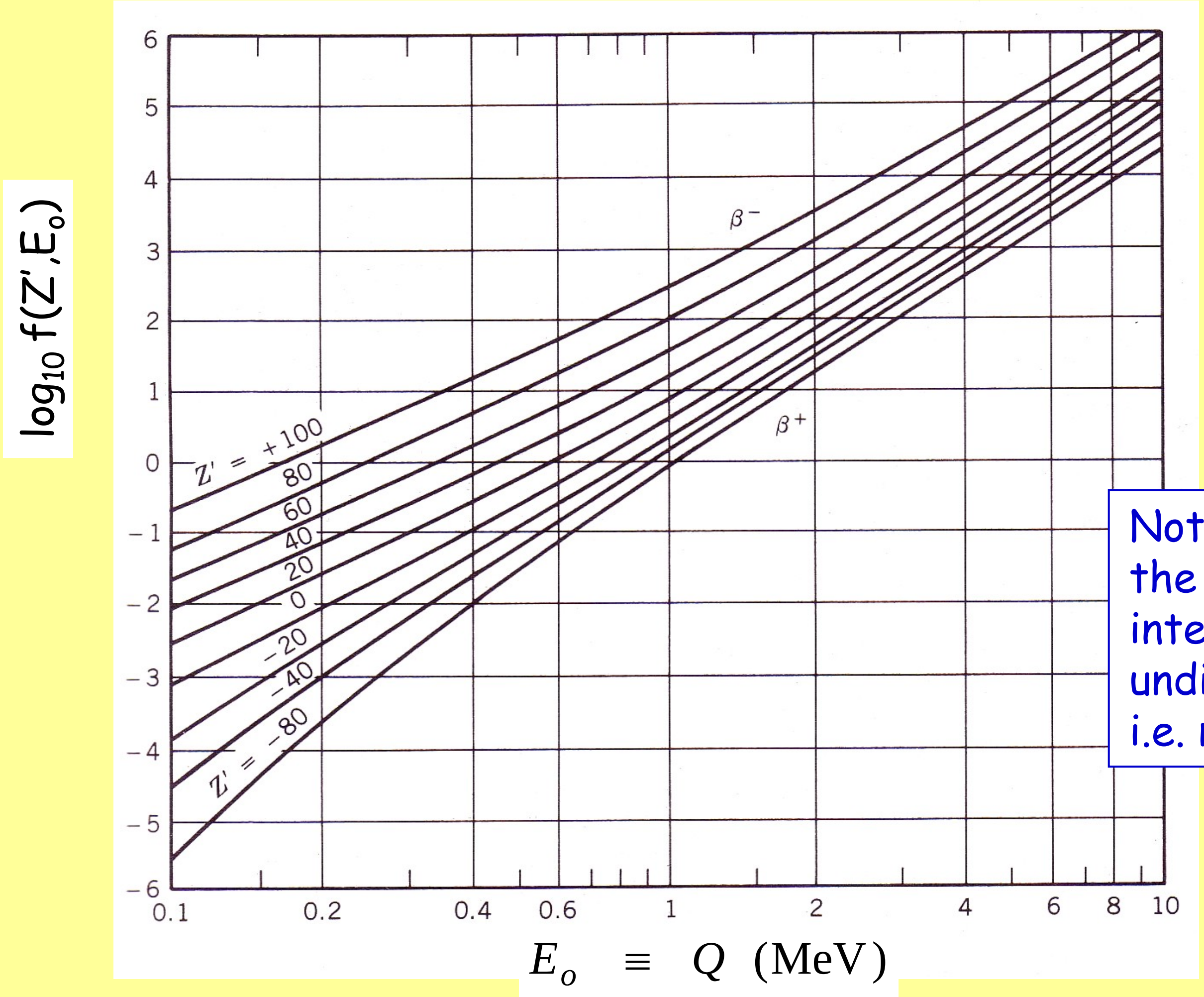
Key point: <u>apart from the nuclear matrix element</u>, the variation in decay rates for different unstable nuclei should <u>only depend on the Fermi integral</u>, which we can calculate independently.

We can use this to test our weak interaction theory!

Krane, Figure 9.8: Dimensionless Fermi integral

By convention:

$$f(Z',E_o) = \frac{1}{m_e^5 c^7} \int_0^{p_{max}} p^2 (E_o - K_e)^2 F(Z',p)\, dp\,, \qquad E_o \equiv Q$$



Note: Z' = 0 gives the "phase space" integral for the undistorted spectrum – i.e. no Coulomb effects.

By convention, the half-life, $t_{1/2} = \tau \ \ln 2 \,, with \ \ \tau = 1/\lambda$ is used as a comparison standard for different nuclear beta decays:

we had:

$$\lambda\,(s^{-1}) \ = \ \frac{G^2}{2\pi^3\hbar^7 c^3} \times \left|M_{nucl}\right|^2 \ \ f(Z',Q)$$

rearranging, we get:

$$f(Z',Q) \ t_{1/2} \ \equiv \ \ f \ t_{1/2} \ = \ \ \ln 2 \ \times \ \frac{2\pi^3\hbar^7}{G^2 m_e^5 c^4 \left|M_{nucl}\right|^2}$$

dimensionless integral from last slide

Notice: The only difference in the "ft" value between different nuclear beta decays is the value of the nuclear matrix element.

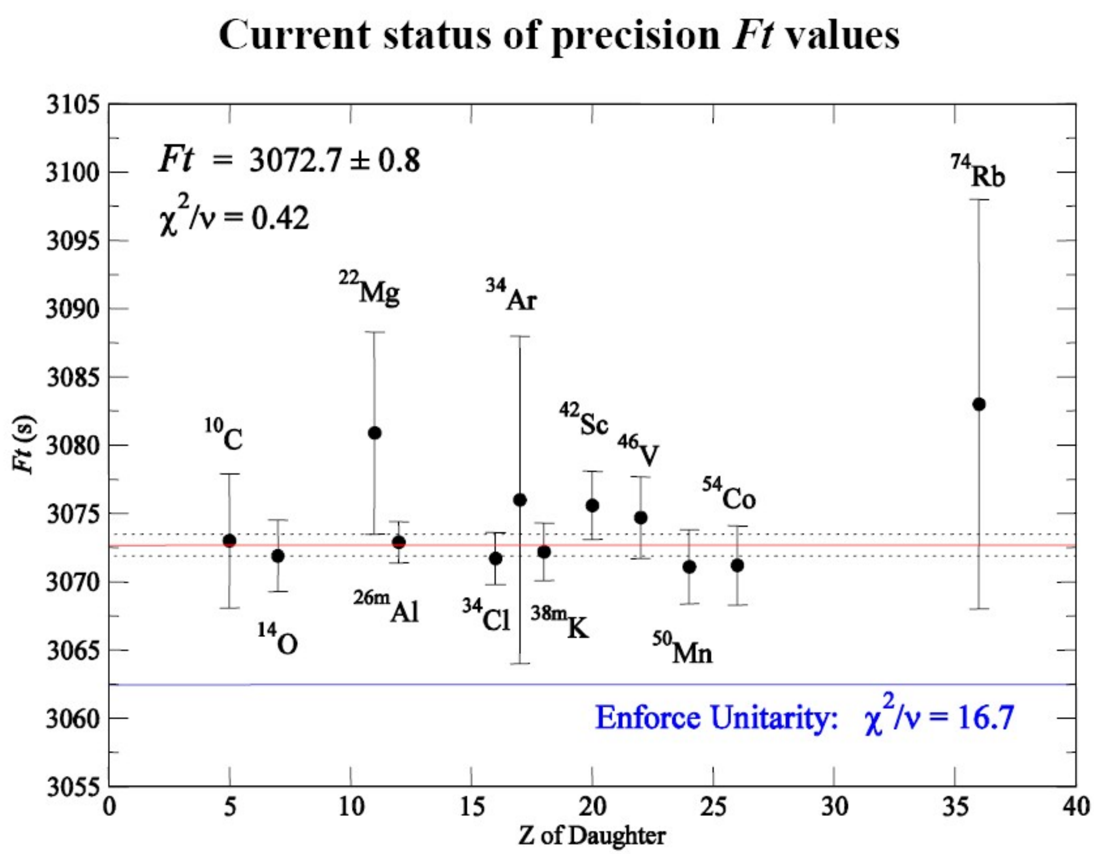
If $|M_{nucl}|^2 = 1$ ("superallowed" case in nuclei), the ft values can be used to determine the weak coupling constants $G = (G_V, G_A)$

Special case: "superallowed" decays in nuclei with initial and final nuclear states 0+ $\rightarrow$ 0+, e.g.

$${}^{14}_{8}\mathrm{O} \ \rightarrow \ {}^{14}_{7}\mathrm{N} + \ e^+ \ + \ \nu_e$$

must have S = 0 for the leptons $\rightarrow$ pure Fermi decay...

Superallowed Fermi 0+ $\rightarrow$ 0+ decays: world's best data for light nuclei



J.C. Hardy and I.S. Towner, PRC **71**, 055501 (2005)

slide by Gordon Ball, TRIUMF – TITAN workshop, 2005

- all have the same ft value ~ 3100 sec
- determines the weak coupling constant for Fermi decays:

$$G_V = (1.1358 \pm 0.0004) \times 10^{-5}\ (\hbar c)^3 / \mathrm{GeV}^2$$

(And G_A/G_V = -1.25, more later....)

Can we understand beta decay rates in general?

first page of Krane, Appendix C: (symbol ε stands for electron capture/ β+ decay)

→ 27 isotopes: 8 β^- decays, 6 β^+ decays, spanning 16 orders of magnitude in rate!

	Z	A	Atomic mass (u)	I^π	*Abundance* or Half-life
H	1	1	1.007825	$\frac{1}{2}^+$	*99.985%*
		2	2.014102	1^+	*0.015%*
		3	3.016049	$\frac{1}{2}^+$	12.3 y (β^-)
He	2	3	3.016029	$\frac{1}{2}^+$	*1.38 × 10⁻⁴%*
		4	4.002603	0^+	*99.99986%*
Li	3	6	6.015121	1^+	*7.5%*
		7	7.016003	$\frac{3}{2}^-$	*92.5%*
		8	8.022486	2^+	0.84 s (β^-)
Be	4	7	7.016928	$\frac{3}{2}^-$	53.3 d (ε)
		8	8.005305	0^+	0.07 fs (α)
		9	9.012182	$\frac{3}{2}^-$	*100%* slowest
		10	10.013534	0^+	1.6 My (β^-)
		11	11.021658	$\frac{1}{2}^+$	13.8 s (β^-)
B	5	8	8.024606	2^+	0.77 s (ε)
		9	9.013329	$\frac{3}{2}^-$	0.85 as (α)
		10	10.012937	3^+	*19.8%*
		11	11.009305	$\frac{3}{2}^-$	*80.2%*
		12	12.014353	1^+	20.4 ms (β^-)
		13	13.017780	$\frac{3}{2}^-$	17.4 ms (β^-)
C	6	9	9.031039	$\frac{3}{2}^-$	0.13 s (ε)
		10	10.016856	0^+	19.2 s (ε)
		11	11.011433	$\frac{3}{2}^-$	20.4 m (ε)
		12	12.000000	0^+	*98.89%*
		13	13.003355	$\frac{1}{2}^-$	*1.11%*
		14	14.003242	0^+	5730 y (β^-)
		15	15.010599	$\frac{1}{2}^+$	2.45 s (β^-) fastest
N	7	12	12.018613	1^+	11 ms (ε)
		13	13.005739	$\frac{1}{2}^-$	9.96 m (ε)
		14	14.003074	1^+	*99.63%*
		15	15.000109	$\frac{1}{2}^-$	*0.366%*
		16	16.006100	2^-	7.13 s (β^-)

1. According to our theory, the very slow decay: (1.6 x 10^6 yrs)

$$ {}^{10}_{4}\mathrm{Be}\,(0^+) \rightarrow {}^{10}_{5}\mathrm{B}\,(3^+) + e^- + \bar{\nu}_e $$

should not occur at all, because angular momentum does not add up, i.e.:

$$ \vec{0} \neq \vec{3} + (\vec{0} \;\; or \;\; \vec{1}) $$

2. Another example: (16.1 hr)

$$ {}^{76}_{35}\mathrm{Br}\,(1^-) \rightarrow {}^{76}_{34}\mathrm{Se}\,(0^+) + e^+ + \nu_e $$

This should not occur because the wavefunctions in the nuclear matrix element have opposite parity, so the integrand is odd and should vanish:

$$ \boxed{M_{nuclear} \equiv \int \psi_f^*(\vec{r}) \;\; \psi_i(\vec{r})\, d^3 r = 0 \quad ???} $$

Forbidden Decays:

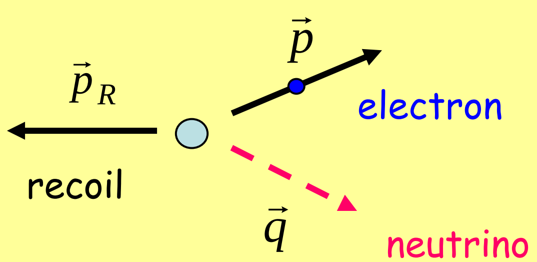
These are two examples of forbidden decays – they cannot proceed under the allowed approximation, since

$$M_{if} = G \int \psi^*_{p,f}(\vec{r})\, \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r})\, \psi_{n,i}(\vec{r})\ d^3r = 0 \quad if \quad \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r}) = \frac{1}{V}$$

Is there some other way they can occur?

Reconsider the electron and antineutrino wave function as a multipole expansion:

$$V\ \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r}) = e^{i\vec{p}_R \bullet \vec{r}/\hbar} \equiv \sum_{L=0}^{\infty} i^L (2L+1)\ j_L(p_R r/\hbar)\ P_L(\cos\theta)$$



$$j_L = spherical \quad Bessel \quad Function$$

$$P_L(\cos\ \theta) = Legendre \quad polynomial$$

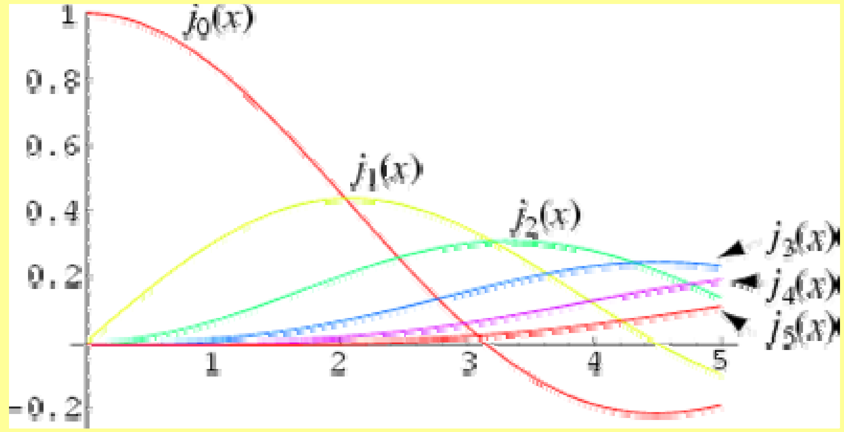
Beyond the "allowed approximation":

$$e^{i\vec{p}_R \bullet \vec{r}/\hbar} \equiv \sum_{L=0}^{\infty} i^L \, (2L+1) \; j_L(p_R r/\hbar) \; P_L \, (\cos\theta)$$

spherical Bessel functions:

$$j_o(x) = \frac{\sin x}{x}; \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \ldots.$$

$$with \;\; x = p_R r/\hbar$$

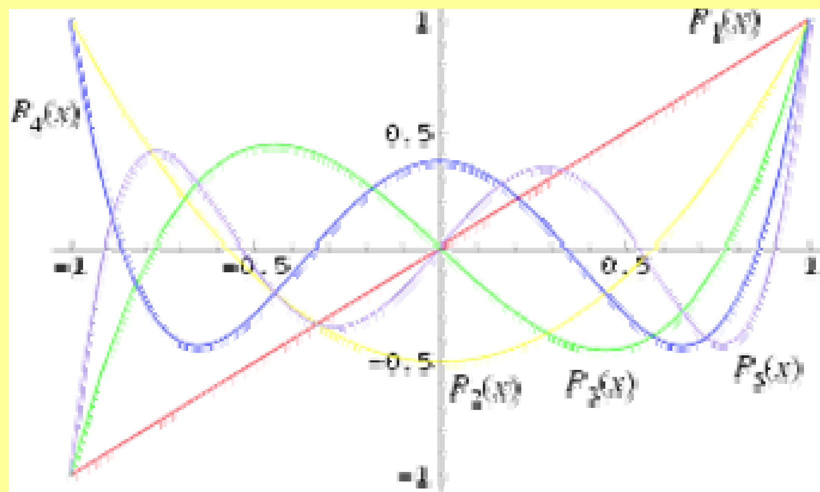


for successively larger L, they become more significant for larger recoil momentum
→ this will change the momentum dependence of our prediction !

Legendre polynomials:

$$P_o = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}(3x^2 - 1) \ldots.$$

$$with \;\; x = \cos\theta$$



these introduce a new angular dependence to the integrand for M_{if} → equivalent to angular momentum L

Effect on beta transitions:

- angular momentum coupling for the multipole order L, together with S and nuclear angular momentum allows previously impossible reactions to proceed

- multipole term has parity $(-1)^L$, which allows nuclear states of opposite parity to be "connected" by the beta decay operator

- momentum dependence of the matrix element varies as $(p_R r/\hbar)^L$ ...

  since this is small, the lowest multipole order L that satisfies the conservation laws will dominate the transition

  rate ~ $|M|^2 \sim (p_R\, r/\hbar)^{2L} \cong (0.01)^{2L} \rightarrow$ dramatically smaller for large L

  momentum dependence also affects the shape of the spectrum; Kurie plots are not linear unless "shape factors" are taken into account....

- naming convention:

L = 0	allowed
L = 1	first forbidden
L = 2	second forbidden
L = 3	third forbidden....

Classification: all known decays....

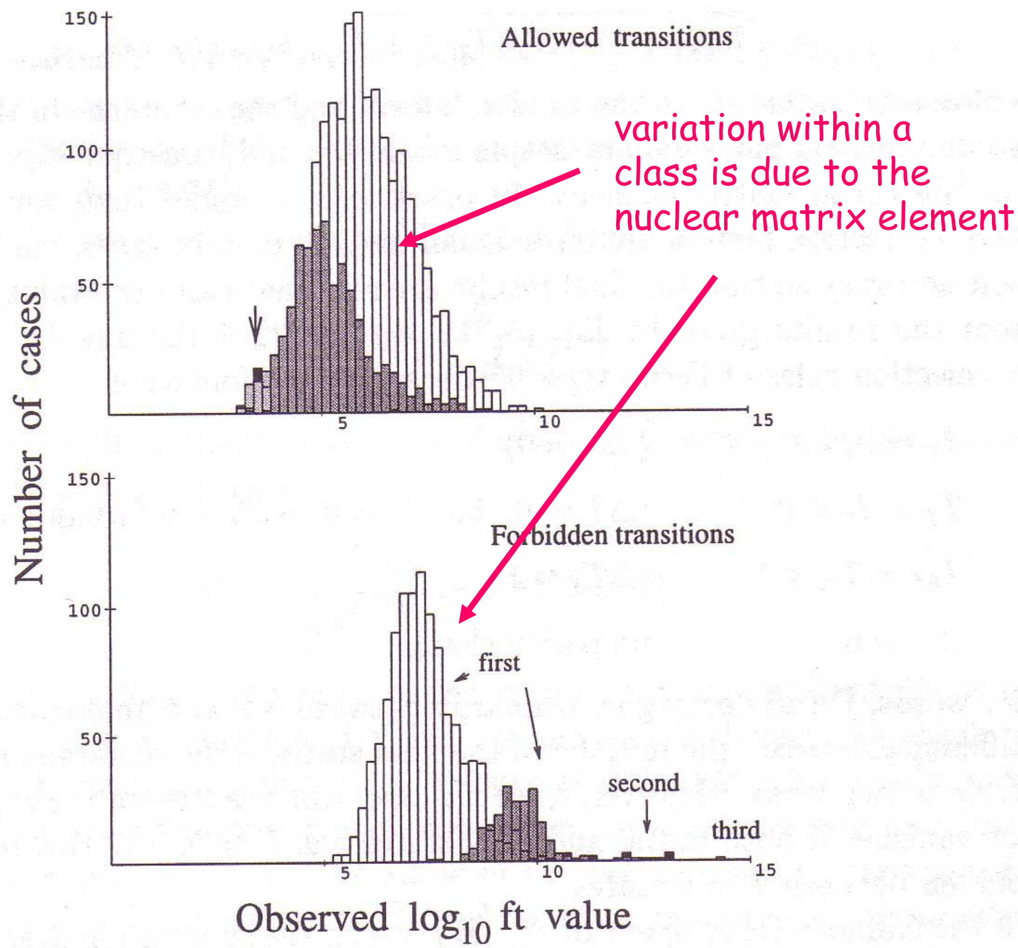
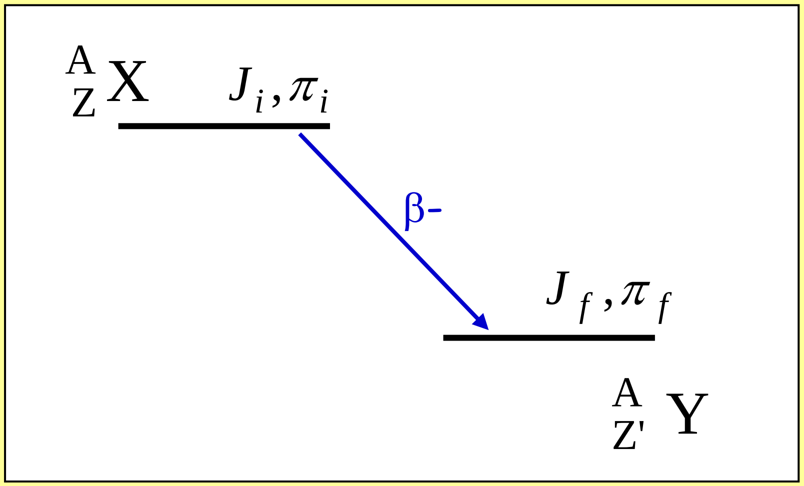
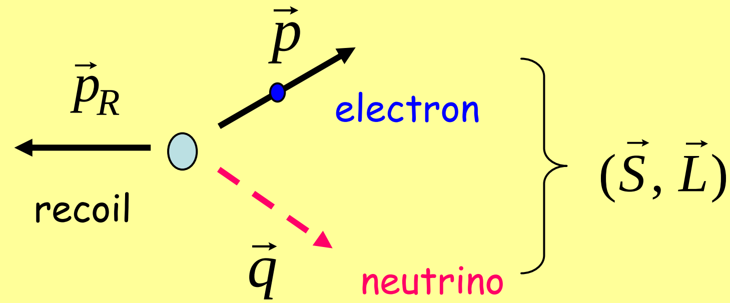


Figure 5-8: Systematics of observed log ft values. The grey area in the upper panel shows 718 cases of $0^+ \rightleftharpoons 1^+$ allowed transitions, and the remaining 1741 cases of other allowed decays are shown by the white histogram. The peak of the distribution for the 24 cases of $0^+ \to 0^+$ superallowed decay is indicated by the arrow.

Classification of decays: general

Nuclear case: $${}^{A}_{Z}X \rightarrow {}^{A}_{Z'}Y + e^{-} + \bar{\nu}_e$$





Conservation laws:

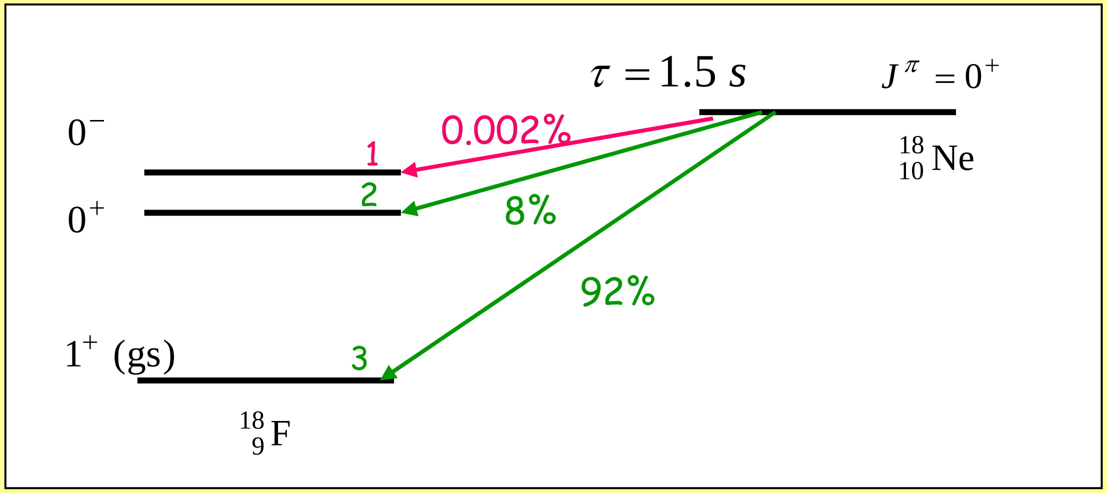
$$\vec{J}_i = \vec{J}_f + \vec{S} + \vec{L}$$

$$\pi_i = \pi_f \, (-1)^L$$

with S = 0 (Fermi) or S = 1 (Gamow-Teller)

Smallest value of L that is consistent with conservation laws will dominate the transition.

Example: $\beta+$ decay of ^{18}Ne



Branching ratio (BR): the fraction of decays that go to a particular final state.

In this case, $\lambda_{total} = 1/\tau = 0.667\ sec^{-1}$; $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, with $\lambda_i = BR(i)\ \lambda_{total}$

Transition 1: **$0^+ \rightarrow 0^-$** This is a **first forbidden GT decay**, with the slowest partial rate:

$$\vec{0} = \vec{0} + \vec{S} + \vec{L}\ ; \quad (+) = (-)\times(-1)^{L} \quad \rightarrow \quad L = 1,\ S = 1$$

Transition 2: $0^+ \rightarrow 0^+$ This is an **allowed Fermi decay**:

$$\vec{0} = \vec{0} + \vec{S} + \vec{L}\ ; \quad (+) = (+)\times(-1)^{L} \quad \rightarrow \quad L = 0,\ S = 0$$

Transition 3: $0^+ \rightarrow 1^+$ This is an **allowed Gamow-Teller decay**

$$\vec{0} = \vec{1} + \vec{S} + \vec{L}\ ; \quad (+) = (+)\times(-1)^{L} \quad \rightarrow \quad L = 0,\ S = 1$$